A NOTE ON THE BERNSTEIN-VON MISES THEOREM FOR A NON-REGULAR SITUATION[1]

by J. D. Borwanker

Technical Report No. 101

University of Minnesota
Minneapolis, Minnesota

[1] Prepared with the support of the U. S. Air Force Grant No. AFOSR-885-65.

1. Introduction and Summary: While discussing the asymptotic properties of the maximum likelihood estimate based on independent and identically distributed chance observations, LeCam [1] has shown that the posterior density of the parameter converges in total variation to a normal density, almost surely with respect to the true distribution. The above result, which we shall refer to as Bernstein-von Mises theorem, has been extended to Markov-dependent random variables by G. Kallianpur and J. D. Borwanker [2]. However, the results in both the papers mentioned above have been obtained under certain regularity conditions. It would be of interest to know what happens if some of the regularity conditions are not satisfied. In the present paper, we consider the case of independent identically distributed random variables whose range depends on parameters.

Let $X_1, X_2,\ldots,X_n$ be i.i.d. random variables with p.d.f. defined as follows:

$$
\begin{aligned}
f(x;\varphi,\theta) &= \frac{h(x)}{H(\varphi,\theta)} && \varphi < x < \theta, \quad -\infty < \varphi < \theta < \infty \\
&= 0 && \text{otherwise},
\end{aligned}
$$

where $h(x) \geq 0$ is a locally integrable function and $H(\varphi,\theta) = \int_\varphi^\theta h(x)dx$, (φ,θ) being two parameters. Let $P_{\varphi,\theta}$ be the product measure determined by $f(x;\varphi,\theta)$. By P_0 we shall mean P_{φ_0,θ_0}, where (φ_0,θ_0) is the true parameter point. Let Λ be a prior probability measure on the (φ,θ) space and F_n the posterior distribution, given Λ and $X_1, X_2,\ldots,X_n$. We show in Theorem 2.1 that F_n converges a.s. P_0 to the degenerate distribution at (φ_0,θ_0). Moreover, if Λ is absolutely continuous with respect to the Lebesgue measure and f_n denotes the posterior density, then we show that with probability P_0 equal to one, f_n converges in variation to the product of two exponential densities (Theorem 2.2).

2. Theorem 2.1: Let $X_1, X_2, \ldots, X_n$ be independent identically distributed random variables with p.d.f. $f(x;\varphi,\theta)$ as defined in Section 1. Set $\bar{\varphi}_n = \min(X_1,\ldots,X_n)$ and $\bar{\theta}_n = \max(X_1,\ldots,X_n)$. Assume that

(a) There exist open neighborhoods $A(\theta_0)$, $A(\varphi_0)$ of θ_0 and φ_0, respectively such that $H(\varphi,\theta)$ is a strictly increasing function of θ for θ in $A(\theta_0)$ and a strictly decreasing function of φ for φ in $A(\varphi_0)$.

Let Λ be a prior distribution on (φ,θ)-space such that

(b) $\displaystyle\int \frac{d\Lambda(\varphi,\theta)}{H^n(\varphi,\theta)} < \infty$ for each n.

(c) $\Lambda\{(\varphi_0 - \epsilon < \varphi \le \varphi_0,\ \ \theta_0 \le \theta < \theta_0 + \epsilon)\} > 0$ for each $\epsilon > 0$.

Define

$$F_n(B) = F_n(B;x_1,\ldots,x_n) = \frac{\displaystyle\int_B \frac{\prod_{i=1}^n h(x_i)\, d\Lambda}{H^n(\varphi,\theta)}}{\displaystyle\int \frac{\prod_{i=1}^n h(x_i)\, d\Lambda}{H^n(\varphi,\theta)}}$$

Then for every open set B

$$P_0\left[F_n(B) \to \begin{cases} 1 & (\varphi_0,\theta_0) \in B \\ 0 & \text{otherwise} \end{cases}\right] = 1.$$

Proof: Without loss of generality, let $A(\theta_0) = (\theta_0-2\epsilon,\ \theta_0+2\epsilon)$, $A(\varphi_0) = (\varphi_0-2\epsilon,\ \varphi_0+2\epsilon)$ and $B = \{\varphi_0-2\epsilon < \varphi < \varphi_0+2\epsilon,\ \ \theta_0-2\epsilon < \theta < \theta_0+2\epsilon\}$ for some fixed $\epsilon > 0$. Note that $(\bar{\varphi}_n,\bar{\theta}_n)$ converges to (φ_0,θ_0) a.s. P_0. This is so because $P_0(\theta_0 - \bar{\theta}_n > \epsilon) = H^n(\varphi_0,\ \theta_0-\epsilon)/H^n(\varphi_0,\theta_0) \to 0$ since $H(\varphi,\theta)$ is a strictly increasing function of θ for θ in $A(\theta_0)$. Hence $\bar{\theta}_n \to \theta_0$ in probability P_0. However, for each sequence

$\underline{x} = (x_1, x_2, \dots)$, $\bar{\theta}_n(\underline{x}) = \max(x_1, \dots, x_n)$ is a nondecreasing sequence of values bounded above by θ_0 a.s. (P_0). Hence $\lim_n \bar{\theta}_n(\underline{x})$ exists a.s. P_0. Call this limit $\theta^*(\underline{x})$. We have

$$P_0(\theta_0 - \theta^* > \epsilon) \begin{cases} \leq P_0[\theta_0 - \bar{\theta}_n > \epsilon] & \text{for all } n, \\ < \delta & \text{for } n > N(\epsilon, \delta). \end{cases}$$

Since ϵ, δ are arbitrary, it follows that $\theta^* = \theta_0$ a.s. P_0. A similar argument shows that $\bar{\varphi}_n \to \varphi_0$ a.s. P_0. Hence for almost all (P_0) $\underline{x}$, there exists an $N_1(\underline{x}, \epsilon)$ such that

$$\text{(2.1)} \qquad \varphi_0 \leq \bar{\varphi}_n < \varphi_0 + \epsilon, \quad \theta_0 - \epsilon < \bar{\theta}_n \leq \theta_0, \qquad \text{for } n > N_1(\underline{x}, \epsilon).$$

Let $L_n(\varphi, \theta; \underline{x}) = L_n(\varphi, \theta; x_1, \dots, x_n)$ denote the likelihood of (φ, θ) given $x_1, \dots, x_n$. We know that

$$\text{(2.2)} \qquad L_n(\varphi, \theta; \underline{x}) = \prod_{i=1}^{n} h(x_i) / H^n(\varphi, \theta) \qquad \text{for } \bar{\varphi}_n \geq \varphi \text{ and } \bar{\theta}_n \leq \theta.$$

Hence

$$F_n(B, \underline{x}) = \int_B L_n(\varphi, \theta)\, d\Lambda \Big/ \int\int L_n(\varphi, \theta)\, d\Lambda$$

$$= \int_{\varphi_0 - 2\epsilon}^{\bar{\varphi}_n} \int_{\bar{\theta}_n}^{\theta_0 + 2\epsilon} \frac{\prod h(x_i)\, d\Lambda}{H^n(\varphi, \theta)} \Bigg/ \int_{\bar{\theta}_n}^{\infty} \int_{-\infty}^{\bar{\varphi}_n} \frac{\prod h(x_i)\, d\Lambda}{H^n(\varphi, \theta)}$$

for $n > N_1(\underline{x}, \epsilon)$. The expression is well defined because of condition (b). Let $d = \min(H(\varphi_0, \theta_0 + \epsilon), H(\varphi_0 - \epsilon, \theta_0))$.

(2.3) Let $\delta = \min(H(\varphi_0, \theta_0 + 2\epsilon, H(\varphi_0 - 2\epsilon, \theta_0)) - d$. Note that $\delta > 0$ because of assumption (a).

(2.4) Define $C_d = \{(\varphi, \theta): H(\varphi, \theta) = d\}$.

Because of the monotonicity property of H (strictly increasing in θ and strictly decreasing in φ) in B, C_d has the property that if (φ_1,θ_1) (φ_2,θ_2) are two points in C_d, then either $\varphi_1 < \varphi_2$, $\theta_1 < \theta_2$ or $\varphi_1 > \varphi_2$, $\theta_1 > \theta_2$. Let D be the subset of B bounded by C_d and the lines $\varphi = \varphi_0$ and $\theta = \theta_0$ so that for all (φ,θ) in D $H(\varphi,\theta) < d$. Moreover condition (b) implies that $\Lambda(D) > 0$. Now

$$\frac{F_n(B^c)}{F_n(B)} \le \frac{F_n(B^c)}{F_n(D)}$$

$$\le \frac{\int_{\bar{\theta}_n}^{\infty}\int_{-\infty}^{\varphi_0-2\epsilon} \frac{d\Lambda}{H^n(\varphi,\theta)} + \int_{\theta_0+2\epsilon}^{\infty}\int_{-\infty}^{\bar{\varphi}_n} \frac{d\Lambda}{H^n(\varphi,\theta)}}{\int_D \frac{d\Lambda}{H^n(\varphi,\theta)}} \quad \text{for} \quad n > N(\underline{x},\epsilon)$$

$$\le \left[\frac{1}{H^n(\varphi_0-2\epsilon,\ \bar{\theta}_n)} + \frac{1}{H^n(\bar{\varphi}_n,\ \theta_0+2\epsilon)}\right] \frac{d^n}{\Lambda(D)} .$$

Choose an $N(\underline{x},\epsilon,H,\delta) \ge N_1(\underline{x},\epsilon)$ such that, for $n > N$,

$$H(\varphi_0-2\epsilon,\ \bar{\theta}_n) > H(\varphi_0-2\epsilon,\ \theta_0) - \delta/2 > d$$

$$H(\bar{\varphi}_n,\ \theta_0+2\epsilon) > H(\varphi_0,\ \theta_0+2\epsilon) - \delta/2 > d.$$

Then

$$\frac{F_n(B^c)}{F_n(B)} < \frac{d^n}{(H(\varphi_0-2\epsilon,\ \theta_0)-\delta/2)^n\Lambda(D)} + \frac{d^n}{(H(\varphi_0,\ \theta_0+2\epsilon)-\delta/2)^n\Lambda(D)}$$

for $n > N$

$\rightarrow 0$, which proves the theorem.

We need the following lemma for the proof of the next theorem (Bernstein-von Mises theorem).

Lemma 2.1: Let the conditions of Theorem 2.1 be satisfied. In addition, let Λ be absolutely continuous with respect to the Lebesgue measure with probability density function $\lambda(\varphi,\theta)$. Set

$$p_n(\varphi,\theta) = L_n(\varphi,\theta)\,\lambda(\varphi,\theta) \Big/ \int\int_B L_n(s,t)\,\lambda(s,t)\,ds\,dt,$$

where B is an open set containing (φ_0,θ_0).

If $f_n(\varphi,\theta)$ denotes the posterior density then

$$\int\int |f_n(\varphi,\theta) - p_n(\varphi,\theta)|\,d\varphi d\theta \to 0 \qquad \text{a.s. } P_0.$$

Proof: It is easy to see that

$$\int\int |f_n(\varphi,\theta) - p_n(\varphi,\theta)|\,d\varphi d\theta \le 2\int_{B^c} \frac{\lambda(\varphi,\theta)}{H^n(\varphi,\theta)}\,d\varphi d\theta \Big/ \int\int \frac{\lambda(\varphi,\theta)}{H^n(\varphi,\theta)}\,d\varphi d\theta$$

which converges to zero a.s. P_0 (by Theorem 2.1).

We now prove the main result, viz; the Bernstein-von Mises Theorem.

Theorem 2.2: Let the conditions of Lemma 2.1 be satisfied. In addition, let the neighborhoods $A(\theta_0)$ and $A(\varphi_0)$ be such that

i) $\frac{d}{du}h(u)$ exists and is bounded (by, say, C_1) for $u \in A(\theta_0)$ and $A(\varphi_0)$.

ii) $\frac{\partial}{\partial\varphi}\lambda(\varphi,\theta)$ and $\frac{\partial}{\partial\theta}\lambda(\varphi,\theta)$ exist and are bounded (say, by C_2), in $A(\varphi_0)\times A(\theta_0)$.

iii) $\frac{\partial}{\partial\varphi}\left[\frac{\lambda(\varphi,\theta)}{h(\varphi)}\right]$ and $\frac{\partial}{\partial\theta}\left[\frac{\lambda(\varphi,\theta)}{h(\theta)}\right]$ exist and are bounded (say, by C_3), in $A(\varphi_0)\times A(\theta_0)$.

Then

$$\int\int |f_n(\varphi,\theta) - q_n(\varphi,\theta)|\,d\varphi d\theta \to 0 \qquad \text{a.s. } P_0,$$

where $q_n(\varphi,\theta) = \frac{n^2 h(\bar\varphi)h(\bar\theta)}{H^2(\bar\varphi_n,\bar\theta_n)} \exp\left[-\frac{n(\bar\varphi-\varphi)\,h(\bar\varphi) - n(\bar\theta-\theta)\,h(\bar\theta)}{H(\bar\varphi_n,\bar\theta_n)}\right].$

Proof: Since h is positive and continuous in $A(\varphi_0)$ and $A(\theta_0)$, there exists an η such that $\{|\theta-\theta_0| < \eta\} \subseteq A(\theta_0)$, $\{|\varphi-\varphi_0| < \eta\} \subseteq A(\varphi_0)$ and

$$(2.5) \qquad \sup_{|\theta-\theta_0|<\eta} |h(\theta) - h(\theta_0)| < \frac{h(\theta_0)}{2}; \quad \sup_{|\varphi-\varphi_0|<\eta} |h(\varphi) - h(\varphi_0)| < \frac{h(\varphi_0)}{2}.$$

Moreover, since λ is positive and continuous in $A(\varphi_0) \times A(\theta_0)$, η can be chosen such that

$$(2.6) \qquad \sup_{\substack{|\theta-\theta_0|<\eta \\ |\varphi-\varphi_0|<\eta}} |\lambda(\varphi,\theta) - \lambda(\varphi_0,\theta_0)| < \frac{\lambda(\varphi_0,\theta_0)}{2}, \qquad \text{and}$$

$$\sup_{\substack{|\theta-\theta_0|<\eta \\ |\varphi-\varphi_0|<\eta}} |H(\varphi,\theta) - H(\varphi_0,\theta_0)| < \frac{H(\varphi_0,\theta_0)}{2}.$$

$$(2.7) \qquad \text{Let } 0 < \epsilon \le \frac{1}{C_3 h(\theta_0)\, h(\varphi_0)} \min\left(\eta, \frac{\lambda(\varphi_0,\theta_0)}{6}\right).$$

Let $B = \{\varphi_0-\epsilon < \varphi < \varphi_0+\epsilon,\ \theta_0-\epsilon < \theta < \theta_0+\epsilon\}$. $B \subseteq A(\varphi_0) \times A(\theta_0)$, for large C_3. Because of Lemma 2.1, it is enough to show that

$$(2.8) \qquad \int\int |p_n(\varphi,\theta) - q_n(\varphi,\theta)|\, d\varphi d\theta \to 0 \qquad \text{a.s. } P_0$$

where p_n is as defined in Lemma 2.1.

Henceforth we shall omit the subscript n in writing $\bar{\varphi}_n$ and $\bar{\theta}_n$.

From (2.1) and (2.2) we know that for $n > N_1(\underline{x},\epsilon)$

$$p_n(\varphi,\theta) = \frac{\lambda(\varphi,\theta)}{H^n(\varphi,\theta)} \int_{\bar{\theta}}^{\theta_0+\epsilon} \int_{\varphi_0-\epsilon}^{\bar{\varphi}} \frac{\lambda(s,t)}{H^n(s,t)}\, ds\, dt.$$

Now

$$\lambda(\varphi,\theta) = \lambda(\bar{\varphi},\bar{\theta}) + \left[(\bar{\varphi}-\varphi)\frac{\partial}{\partial\varphi} + (\theta-\bar{\theta})\frac{\partial}{\partial\theta}\right] \lambda(z_1 z_2)$$

and

$$\int_{\bar\theta}^{\theta_0+\epsilon}\int_{\varphi_0-\epsilon}^{\bar\varphi} \frac{\lambda(s,t)}{H^n(s,t)}\, dsdt = \int_{\bar\theta}^{\theta_0+\epsilon}\int_{\varphi_0-\epsilon}^{\bar\varphi} \frac{h(s)h(t)}{H^n(s,t)}\frac{\lambda(s,t)}{h(s)h(t)}\, dsdt$$

$$= \int_{\bar\theta}^{\theta_0+\epsilon}\int_{\varphi_0-\epsilon}^{\bar\varphi} \frac{h(s)h(t)}{H^n(s,t)} \left[\frac{\lambda(\bar\varphi,\bar\theta)}{h(\bar\varphi)h(\bar\theta)} + \{(\bar\varphi-s)\frac{\partial}{\partial s} + (t-\bar\theta)\frac{\partial}{\partial t}\}\left(\frac{\lambda(z_3,z_4)}{h(z_3)h(z_4)}\right)\right] dsdt$$

(where, in the above expressions, $\varphi \le z_1 \le \bar\varphi$, $\bar\theta \le z_2 \le \theta$, $s \le z_3 \le \bar\varphi$, $\bar\theta \le z_4 \le t$)

$$= \frac{1}{(n-1)(n-2)} G_{n-2}(\bar\varphi,\varphi_0-\epsilon,\ \bar\theta,\theta_0+\epsilon) \left[\frac{\lambda(\bar\varphi,\bar\theta)}{h(\bar\varphi)h(\bar\theta)} + R_n\right]$$

where

(2.9) $$R_n = \frac{(n-1)(n-2)}{G_n(\bar\varphi,\varphi_0-\epsilon,\ \bar\theta,\theta_0+\epsilon)} \int_{\bar\theta}^{\theta_0+\epsilon}\int_{\varphi_0-\epsilon}^{\bar\varphi} \{(\bar\varphi-s)\frac{\partial}{\partial s} + (t-\bar\theta)\frac{\partial}{\partial t}\}\left(\frac{\lambda(z_3,z_4)}{h(z_3)h(z_4)}\right)\frac{h(s)h(t)}{H^n(s,t)}\, dsdt$$

and

$$G_n(\bar\varphi,\varphi_0-\epsilon,\ \bar\theta,\theta_0+\epsilon) = \frac{1}{H^n(\bar\varphi,\bar\theta)} - \frac{1}{H^n(\varphi_0-\epsilon,\bar\theta)} + \frac{1}{H^n(\varphi_0-\epsilon,\theta_0+\epsilon)} - \frac{1}{H^n(\bar\varphi,\theta_0+\epsilon)}.$$

(2.10) $$|R_n| \le \frac{(n-1)(n-2)}{G_n(\bar\varphi,\varphi_0-\epsilon,\ \bar\theta,\theta_0+\epsilon)} \int_{\bar\theta}^{\theta_0+\epsilon}\int_{\varphi_0-\epsilon}^{\bar\varphi} (2\epsilon C_3 + 2\epsilon C_3)\frac{h(s)h(t)}{H^n(s,t)}\, dsdt$$

$$= 4\,\epsilon\, C_3.$$

Therefore

(2.11) $$p_n(\varphi,\theta) = \frac{(n-1)(n-2)\, h(\bar\varphi)\, h(\bar\theta)}{G_{n-2}(\bar\varphi,\varphi_0-\epsilon,\ \bar\theta,\theta_0+\epsilon)\, H^n(\varphi,\theta)}\,[1 + S_n]$$

where

$$S_n = \frac{\{(\bar{\varphi}-\varphi)\frac{\partial}{\partial\varphi} + (\theta-\bar{\theta})\frac{\partial}{\partial\theta}\}\ \lambda(z_1,z_2) - h(\bar{\varphi})h(\bar{\theta})\ R_n}{\lambda(\bar{\varphi},\bar{\theta}) + h(\bar{\varphi})\ h(\bar{\theta})\ R_n}\ .$$

Now

$$(2.12)\qquad \int\int |p_n(\varphi,\theta) - q_n(\varphi,\theta)|\ d\varphi d\theta$$
$$= \int\int_{E_n} q_n(\varphi,\theta)\ d\varphi d\theta + \int_{\bar{\theta}}^{\theta_0+\epsilon} \int_{\varphi_0-\epsilon}^{\bar{\varphi}} |p_n(\varphi,\theta) - q_n(\varphi,\theta)|\ d\varphi d\theta,$$

where E_n is the complement of the set $\{\varphi_0-\epsilon < \varphi < \bar{\varphi},\ \ \bar{\theta} < \theta < \theta_0+\epsilon\}$.

However

$$(2.13)\qquad \int_{E_n} q_n(\varphi,\theta)\ d\varphi d\theta = 1 - \int_{\bar{\theta}}^{\theta_0+\epsilon} \int_{\varphi_0-\epsilon}^{\bar{\varphi}} q_n(\varphi,\theta)\ d\varphi d\theta$$

$$= 1 - [1 - \exp\{\frac{-n(\theta_0+\epsilon-\bar{\theta})\ h(\bar{\theta})}{H(\bar{\varphi},\bar{\theta})}\}]$$

$$[1 - \exp\{\frac{-n(\bar{\varphi}-\varphi_0+\epsilon)\ h(\bar{\varphi})}{H(\bar{\varphi},\bar{\theta})}$$

$$\to 0 \qquad \text{as} \quad n \to \infty.$$

Thus, it is enough to show that the second integral on the right side of (2.12) goes to zero. We have

$$(2.14)\qquad \int_{\bar{\theta}}^{\theta_0+\epsilon} \int_{\varphi_0-\epsilon}^{\bar{\varphi}} |p_n(\varphi,\theta) - q_n(\varphi,\theta)|\ d\varphi d\theta$$

$$\leq \int_{\bar{\theta}}^{\bar{\theta}+A/n} \int_{\bar{\varphi}-A/n}^{\bar{\varphi}} \left| \frac{(n-1)(n-2)\ h(\bar{\theta})\ h(\bar{\varphi})}{G_{n-2}(\bar{\varphi},\varphi_0-\epsilon,\ \bar{\theta},\theta_0+\epsilon)\ H^n(\varphi,\theta)} - q_n(\varphi,\theta) \right| d\varphi d\theta$$

$$+ \int_{\bar{\theta}+A/n}^{\theta_0+\epsilon} \int_{\varphi_0-\epsilon}^{\bar{\varphi}} q_n(\varphi,\theta)\ d\varphi d\theta + \int_{\bar{\theta}}^{\theta_0+\epsilon} \int_{\varphi_0-\epsilon}^{\bar{\varphi}-A/n} q_n(\varphi,\theta)\ d\varphi d\theta$$

$$+ \int_{\bar{\theta}}^{\theta_0+\epsilon} \int_{\varphi_0-\epsilon}^{\bar{\varphi}} \frac{(n-1)(n-2)h(\bar{\varphi})h(\bar{\theta})\ S_n\ d\varphi d\theta}{G_{n-2}(\bar{\varphi},\varphi_0-\epsilon,\ \bar{\theta},\theta_0+\epsilon)\ H^n(\varphi,\theta)}$$

$$+ \int_{\bar\theta}^{\theta_0+\epsilon} \int_{\varphi_0-\epsilon}^{\bar\varphi-A/n} \frac{(n-1)(n-2)\, h(\bar\theta)\, h(\bar\varphi)}{G_{n-2}(\bar\varphi,\varphi_0-\epsilon,\ \bar\theta,\theta_0+\epsilon)\ H^n(\varphi,\theta)}\ d\varphi d\theta$$

$$+ \int_{\bar\theta+A/n}^{\theta_0+\epsilon} \int_{\varphi_0-\epsilon}^{\bar\varphi} \frac{(n-1)(n-2)\, h(\bar\theta)\, h(\bar\varphi)}{G_{n-2}(\bar\varphi,\varphi_0-\epsilon,\ \bar\theta,\theta_0+\epsilon)\ H^n(\varphi,\theta)}\ d\varphi d\theta .$$

The last integral

(2.15) $$\le \frac{4}{h(\varphi_0)h(\theta_0)} \int_{\bar\theta+A/n}^{\theta_0+\epsilon} \int_{\varphi_0-\epsilon}^{\bar\varphi} \frac{(n-1)(n-2)\, h(\bar\theta)\, h(\bar\varphi)\, h(\varphi)\, h(\theta)}{G_{n-2}(\bar\varphi,\varphi_0-\epsilon,\ \bar\theta,\theta_0+\epsilon)\ H^n(\varphi,\theta)}\ d\varphi d\theta$$

by (2.5)

$$= \frac{4}{h(\varphi_0)h(\theta_0)}\ \frac{G_{n-2}(\bar\varphi,\varphi_0-\epsilon,\ \bar\theta+A/n,\ \theta_0+\epsilon)}{G_{n-2}(\bar\varphi,\varphi_0-\epsilon,\ \bar\theta,\theta_0+\epsilon)}$$

$$\simeq \frac{4}{h(\varphi_0)h(\theta_0)}\ \frac{H^{n-2}(\bar\varphi,\bar\theta)}{H^{n-2}(\bar\varphi,\bar\theta+A/n)}\ ,$$

by the definition of G and strict monotnicity of H. But

$$H^{n-2}(\bar\varphi,\bar\theta+A/n) = H^n(\bar\varphi,\bar\theta)\ [1 + (A/n)h(z_5)]^n \qquad \bar\theta \le z_5 \le \bar\theta+A/n$$

$$\ge H^n(\bar\varphi,\bar\theta)\ [1 + \frac{Ah(\theta_n)}{2n}]^n.$$

Hence, for large n and some constant α, (2.15) is

$$\le \frac{4\alpha}{h(\theta_0)h(\varphi_0)}\ [1 + \frac{Ah(\theta_0)}{2n}]^n$$

$$\to \frac{4\alpha}{h(\varphi_0)h(\theta_0)}\ \exp\ [- \frac{Ah(\theta_0)}{2}].$$

(2.16) Similarly, the fifth integral can be shown to be less than

$$\frac{4\alpha}{h(\varphi_0)h(\theta_0)}\ \exp\ [- \frac{Ah(\varphi_0)}{2}].$$

The second term on the right side of (2.14)

$$(2.17) \qquad \leq \exp\left[\frac{-Ah(\theta_0)}{3H(\varphi_0,\theta_0)}\right] \qquad \text{by (2.6).}$$

Similarly, the third term on the right side of (2.14)

$$(2.18) \qquad \leq \exp\left[\frac{-Ah(\varphi_0)}{3H(\varphi_0,\theta_0)}\right] \qquad \text{by (2.6).}$$

'A' can be chosen large enough to make expressions in (2.15) to (2.18) $< \epsilon$. The fourth expression on the right side of (2.14) is

$$\leq |S_n|\, h(\bar{\varphi})h(\bar{\theta}) \int_{\bar{\theta}}^{\theta_0+\epsilon} \int_{\varphi_0-\epsilon}^{\bar{\bar{\varphi}}} \frac{h(\varphi)h(\theta)\ (n-1)(n-2)\ d\varphi d\theta}{G_{n-2}(\bar{\varphi},\varphi_0-\epsilon,\ \bar{\theta},\theta_0+\epsilon)\ H^n(\varphi,\theta)\ h(\varphi)h(\theta)}$$

$$\leq \frac{(2\epsilon C_2 + 2\epsilon C_2) + 9h(\theta_0)h(\varphi_0)\ \epsilon\ C_3}{(1/2)\lambda(\varphi_0,\theta_0) - (9/4)h(\varphi_0)h(\theta_0)\ \epsilon\ C_3} \quad \frac{9h(\theta_0)\ h(\varphi_0)}{h(\varphi_0)h(\theta_0)}$$

(by (2.5), (2.6))

$$(2.19) \qquad \leq \frac{8(4C_2 + 9h(\theta_0)h(\varphi_0)C_3)9\epsilon}{\lambda(\varphi_0,\theta_0)} \qquad \text{(by (2.5), (2.7)).}$$

Lastly, by putting $\alpha = n(\bar{\varphi}-\varphi)$, $\beta = n(\theta-\bar{\theta})$, the first term on the right side of (2.14) is

$$= \int_0^A \int_0^A \left| \frac{(n-1)(n-2)\ h(\bar{\varphi})\ h(\bar{\theta})}{n^2 G_{n-2}(\bar{\varphi},\varphi_0-\epsilon,\ \bar{\theta},\theta_0+\epsilon)\ H^n(\bar{\varphi}-\frac{\alpha}{n},\ \bar{\theta}+\frac{\beta}{n})} - \frac{h(\bar{\varphi})h(\bar{\theta})}{H^2(\bar{\varphi},\bar{\theta})} \exp\left[\frac{-\alpha h(\varphi) - \beta h(\theta)}{H(\bar{\varphi},\bar{\theta})}\right] \right| d\alpha d\beta$$

$$(2.20) \qquad = \int_0^A \int_0^A \left| \frac{(n-1)(n-2)}{n^2} \frac{h(\bar{\varphi})h(\bar{\theta})}{G_{n-2}} \exp\left[-n \log \left(H(\bar{\varphi},\bar{\theta}) + \frac{\alpha}{n}h(\bar{\varphi}) + \frac{\beta}{n}h(\bar{\theta}) + 0(\frac{1}{n^2})\right)\right] - \frac{h(\bar{\varphi})h(\bar{\theta})}{H^2(\bar{\varphi},\bar{\theta})} \exp\left[\frac{-\alpha h(\bar{\varphi}) - \beta h(\bar{\theta})}{H(\bar{\varphi},\bar{\theta})}\right] \right| d\alpha d\beta.$$

Since α and β are both bounded, for large n,

$$n \log [H(\bar{\varphi},\bar{\theta}) + \frac{\alpha}{n} h(\bar{\varphi}) + \frac{\beta}{n} h(\bar{\theta}) + 0(\frac{1}{n^2})]$$

$$\simeq n[\frac{\alpha}{n} \frac{h(\bar{\varphi})}{H(\bar{\varphi},\bar{\theta})} + \frac{\beta}{n} \frac{h(\bar{\theta})}{H(\bar{\varphi},\bar{\theta})} + 0(\frac{1}{n^2})] + n \log H(\bar{\varphi},\bar{\theta}).$$

Hence the expression in (2.20)

$$(2.21) \qquad \simeq \int_0^A \int_0^A h(\bar{\varphi})h(\bar{\theta}) \exp\left[\frac{-\alpha h(\bar{\varphi}) - \beta h(\bar{\theta})}{H(\bar{\varphi},\bar{\theta})}\right] \left| \frac{(n-1)\ (n-2)}{n^2 G_{n-2} H^n(\bar{\varphi},\bar{\theta})}(1+0(\frac{1}{n})) - \frac{1}{H^2(\bar{\varphi},\bar{\theta})} \right| d\alpha d\beta.$$

Now

$$G_{n-2}(\bar{\varphi},\varphi_0-\epsilon,\ \bar{\theta},\theta_0+\epsilon)\ H^{n-2}(\bar{\varphi},\bar{\theta})$$

$$= \frac{H^{n-2}(\bar{\varphi},\bar{\theta})}{H^{n-2}(\bar{\varphi},\bar{\theta})} - \frac{H^{n-2}(\bar{\varphi},\bar{\theta})}{H^{n-2}(\varphi_0-\epsilon,\bar{\theta})} + \frac{H^{n-2}(\bar{\varphi},\bar{\theta})}{H^{n-2}(\varphi_0-\epsilon,\theta_0+\epsilon)} - \frac{H^{n-2}(\bar{\varphi},\bar{\theta})}{H^{n-2}(\varphi_0-\epsilon,\bar{\theta})}$$

$$\to 1 - 0 + 0 - 0 \qquad \text{as} \quad n \to \infty.$$

Hence (2.21) is equal to

$$\int_0^A \int_0^A \frac{h(\bar{\varphi})h(\bar{\theta})}{H^2(\bar{\varphi},\bar{\theta})} \exp\left[\frac{-\alpha h(\bar{\varphi}) - \beta h(\bar{\theta})}{H(\bar{\varphi},\bar{\theta})}\right] \left|(1 \pm \epsilon)(1 + 0(\frac{1}{n}) - 1\right| d\alpha d\beta$$

$$(2.22) \qquad < 2\epsilon \qquad \text{for} \quad n > N \text{ (say)}.$$

Thus by (2.13), (2.15)to(2.19), (2.22) and Lemma 2.1, we get

$$\int\int |f_n(\varphi,\theta) - q_n(\varphi,\theta)|\, d\varphi d\theta < k\epsilon \qquad \text{for} \quad n > N$$

where k is a constant depending on $\varphi_0, \theta_0, \lambda$ and h. Since ϵ can be chosen arbitrarily small, the theorem is proven.

Corollary: Let $h(x)$ be bounded away from zero for all x and let the conditions of the theorem hold for all φ and θ $(\varphi < \theta)$ instead of just the neighborhoods $A(\varphi_0)$ and $A(\theta_0)$ of φ_0 and θ_0, respectively. Set $\alpha = n(\bar{\bar{\varphi}}-\varphi)$ and $\beta = n(\theta-\bar{\bar{\theta}})$. If $f_n(\alpha,\beta)$ denotes the posterior density of (α,β) given $X_1,\ldots,X_n$, then for every fixed K_1 and K_2

$$f_n(\alpha,\beta) = \frac{h(\bar{\bar{\varphi}})h(\bar{\bar{\theta}})}{H^2(\bar{\bar{\varphi}},\bar{\bar{\theta}})}\{\exp\left[\frac{-\alpha h(\bar{\bar{\varphi}}) - \beta h(\bar{\bar{\theta}})}{h(\bar{\bar{\varphi}},\bar{\bar{\theta}})}\right]\}\,(1 + 0(\tfrac{1}{n})(\alpha + \beta))$$

$$\text{for} \quad 0 < \alpha < K_1, \quad 0 < \beta < K_2.$$

It may be mentioned here that the uniform distribution $(h(x) = 1$ for $\varphi < x < \theta)$ is a special case of the situation discussed above. Actually the theorem would be applicable to truncated distributions (with only truncation points as parameters) for a suitable prior distribution. The one dimensional cases (where $\theta > 0$ and φ is known to be zero or $-\theta$) follow a similar pattern of proof except that q_n is replaced by the corresponding one dimensional density.

My thanks are due to Professor G. Kallianpur for suggesting the problem and various other helpful suggestions.

REFERENCES

[1] LeCam, L. On some asymptotic properties of maximum likelihood estimates, and related Bayes estimates. University of California Publication, (1953).

[2] Kallianpur, G. and Borwanker, J. D. "The Bernstein-von Mises Theorem for Markov Processes." Technical Report No. 96, University of Minnesota, Minneapolis, Minnesota 55455, (submitted for publication).